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THE CORE OF AN ECONOMY WITH NONCONVEX PREFERENCES

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PREFACE

In this Memorandum the authors present some mathematical results concerning a competitive situation arising in economic theory, using the techniques of the theory of games. Game theory is important in its general applicability to a variety of conflict situations—economic, political, and military.

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SUMMARY

A model of a pure exchange economy is investigated without the usual assumption of convex preference sets for the participating traders. The concept of *core*, taken from the theory of games, is applied to show that if there are sufficiently many participants, the economy as a whole will possess a solution that is sociologically stable—i.e., that cannot be upset by any coalition of traders.

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THE CORE OF AN ECONOMY WITH NONCONVEX PREFERENCES

1. INTRODUCTION

In his study of bilateral monopoly, Edgeworth [8] suggested as a fundamental solution concept the now well-known contract curve, consisting of those Pareto-optimal allocations that are at least as attractive to each monopolist as the initial, "no-trade" position. He then pointed out that if the number of traders on each side of the market were increased, the contract curve would under appropriate conditions shrink down to the set of competitive allocations, which are the ones that can be derived from the initial position by direct budgetary optimization by the individual entrepreneurs, under a fixed schedule of prices.

This shrinking of the contract curve as the size of the market increases was based on the following consideration: that no proposed allocation of goods would be finally acceptable to the market if there were a subset of traders who, by recontracting amongst themselves, could do uniformly better. This principle corresponds quite well to the idea of domination, which underlies two important solution concepts in n-person cooperative game theory, namely, the core and the von Neumann-Morgenstern solution.[1] Indeed, if

[1]The core of a game is the set of all undominated outcomes. A solution is any set of outcomes, mutually undominating, that collectively dominate all others. The term "core" was introduced by Gillies and Shapley during

the expanded Edgeworth model, with n producers and n consumers, is regarded as a 2n-person game, the contract curve, curtailed by the recontracting principle, is precisely the core. The uncurtailed contract curve, on the other hand, is a von Neumann-Morgenstern solution— generally not the only one. In the case of bilateral monopoly (n = 1), the core, the contract curve, and the unique von Neumann-Morgenstern solution all coincide.

A number of refinements and extensions of Edgeworth's convergence theorem have been obtained recently by Shubik [18], Scarf [16], Debreu [6], Debreu and Scarf [7], and Aumann [2], who exploit to a greater or lesser extent the game-theoretic point of view. So far, the assumption has always been made that the preferences of the individual traders are convex, since it is well known that without convexity there may be no competitive allocations. Recent articles by Farrell [9], Rothenberg [15], Koopmans [12], and Bator [5] have focussed new attention on the implications of nonconvex preferences. With or without convexity, the existence of a competitive allocation implies the existence of a core, but the core may and often does exist even in the absence of a competitive allocation. For example, a

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a study of the properties of the von Neumann-Morgenstern solutions; see [13], [10]. The core as an independent solution concept was developed by Shapley in lectures at Princeton University, in the fall of 1953.

core always exists for bilateral monopoly. Thus, the core is a more basic economic concept than the competitive equilibrium.

Our object in this paper is to show that even when the core itself does not exist, owing to nonconvexity of preferences, it nevertheless lies "just below the surface," with its location revealed by the presence of certain quasi-cores. These are shown to exist whenever the number of traders in the market is sufficiently large. In the limit, they shrink down to the set of competitive allocations of a "convexified" version of the original model. The corresponding price schedules represent pseudo-equilibria in the original model, at which the excess demand for each good is either zero or else indeterminate in sign, meaning that two sets of individual optimizations can be found that create, respectively, a shortage and a surplus in that good. Thus, a tâtonnement, or other dynamic process, that would converge to the competitive equilibrium if preferences were convex, might be expected in the nonconvex case to exhibit convergence in prices, even though no feasible allocation might exist at the limit.

Generally speaking, our thesis is that nonconvexity in the preference sets is of small consequence when the number of individuals in the economy is large. This is of course not a new observation, but we give it here a mathematically precise expression in a fresh context.

2. ON THE USE OF TRANSFERABLE UTILITIES

The mathematical results that we shall present are to a certain extent illustrative rather than comprehensive, since they rest upon two fairly drastic assumptions in an otherwise general setting. This is primarily a matter of technical convenience: the assumptions are indeed crucial to our method, but not, we believe, to the essential idea embodied in the results.

The first special assumption is that all individuals in the economy have identical preferences, expressed by means of a common, cardinal utility function. The second —and more controversial—is that "utility" itself is freely transferable between individuals. Some discussion of this point is in order.

It has long been the fashion to formulate elementary models of exchange on a barter basis. No Marshallian "utility money," serving simultaneously as a measure of value and a medium of exchange, is postulated. Individual utility functions are often avoided entirely; when they are admitted, they are only occasionally risk-linear, rarely interpersonally comparable, and never transferable.[2] Such elementary barter-based models are relatively easy to think about, and it must be conceded that the suppression of

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[2]To be sure, money-like vehicles for side payments can sometimes be discovered among the goods in exchange, but the conditions imposed thereon—nonnegativity, strict convexity, saturation, etc.—commonly prevent their use as direct comparators and carriers of utility.

money phenomena makes for technical simplicity and lends an air of abstractness and generality to the formulation.

A capsule rationalization of this approach might go as follows: (a) money is not basic to the phenomenon of exchange; (b) when it is found in actual markets, its presence is due to external factors; (c) it brings in many nonelementary complications (e.g., credit, interest, currency backing) that respond better to separate analytical treatment; (d) the money:utility relationship has obvious and celebrated nonlinearities. Build a theory of barter economics first, we are urged, and add money later along with other refinements and elaborations.

When economic theory comes into contact with the theory of games, which regards individuals as independent, resourceful, sophisticated decision-makers rather than elements in a mechanical (albeit "rational") process, then the foregoing doctrine takes a curious turn. Mathematical convenience, which may or may not be a virtue in itself, is now distinctly on the side of transferable utility. "Utility money" smooths away many bristling difficulties that arise when one attempts to deal with a large number of uncoordinated centers of independent strategic choice. With "unrestricted side payments" (the game-theorist's term), the modeling is simplified and many concepts are made clearer, though

a few may be obscured. Moreover, the technical analysis is almost always easier, and the interpretation of the solutions more direct and intuitive, than in the corresponding nontransferable case.

Expediency, then, prompts us to retrace the argument, to see whether a counterrationalization might not also be possible. In this spirit, then, we might argue: (a) that the impulse to compare and exchange utility is a deep-seated expression of a fact of nature, inhering in the market situation itself; (b) that when "utility money" does not exist, rational individuals at once set about to create it; (c) that in a situation where both goods and information are already freely transferable, it would be difficult in fact, and hence unrealistic in a model, to <u>prohibit</u> the creation of a money substitute of some kind, if the traders felt they had use for it. From this point of view, the basic and most elementary models of exchange would properly postulate a freely transferable utility. Later refinements would then examine the effects of various restrictions on transfer, or of anomalies in the money:utility relationship. Pure barter might perhaps be the least interesting, extreme case.

This sketchy discussion certainly does not pretend to settle the question, nor do the authors regard themselves

as committed wholly either to one philosophy or the other. We merely wish to suggest to the reader that our present hypothesis of transferable utility is not only expedient, but methodologically respectable, and perhaps even fundamentally superior to the opposite assumption.

3. AN EXAMPLE[3]

A simple example, with n identical traders, will serve to illustrate the existence problem for cores and competitive equilibria as a function of n, and the rôle played therein by homogeneity and transferability of utility. The nonconvex preferences arise out of <u>preferred ratios</u> in the consumption process, as shown in the indifference map of Fig. 1.[4] The numerical utility function is assumed to be homogeneous of degree one:

$$(1) \qquad U(x) = \max\,[\min(2x_1,\ x_2),\ \min(x_1,\ 2x_2)].$$

There are thus two goods, say gin and tonic, and each traders enters the market with a supply of one unit of each. Each trader is indifferent as between weak drinks (1 to 2) and strong drinks (2 to 1), but he will not take both, and rejects intermediate (or more extreme) concoctions. Utility will be nontransferable except where noted.

[3]This example was described briefly in [7].

[4]The polygonal form of the indifference curves is inessential: the corners can be rounded off and the straight edges bowed. Also, the "notch" can be made as small as we please, by adjusting the coefficients in (1) (compare (28) in Appendix 1).

In looking for a competitive equilibrium, we first observe that unequal prices will never work, since the traders would all want to buy the cheaper good and sell the other. Making utility transferable would do nothing to relieve this imbalance.

With equal prices, however, a competitive allocation can be reached, provided that n is _even_. In fact, each man can exchange 1/3 units of one good for an equal amount of the other, and end up with an efficient bundle—either (4/3, 2/3) or (2/3, 4/3). The payoff is therefore 4/3 "utils" to everyone. Of course it may require cooperative action, or a determined hostess, to settle who is to get what drink!

If n is _odd_, on the other hand, no series of exchanges (at equal prices) can possibly give everyone an efficient ratio, and there is no competitive equilibrium. Rather, we have a pseudo-equilibrium, where the excess demand for either commodity (at equal prices) can be construed as being either positive or negative, but not zero.

Transferable utility changes the picture somewhat. In the _even_ case, the competitive prices and payoffs still exist, and are the same as before, but the allocative possibilities are opened up. Because of the homogeneity of (1), some of the traders may liquidate their holdings for cash, provided that others spend an equal amount to increase their comsumption. In the _odd_ case, this same option works

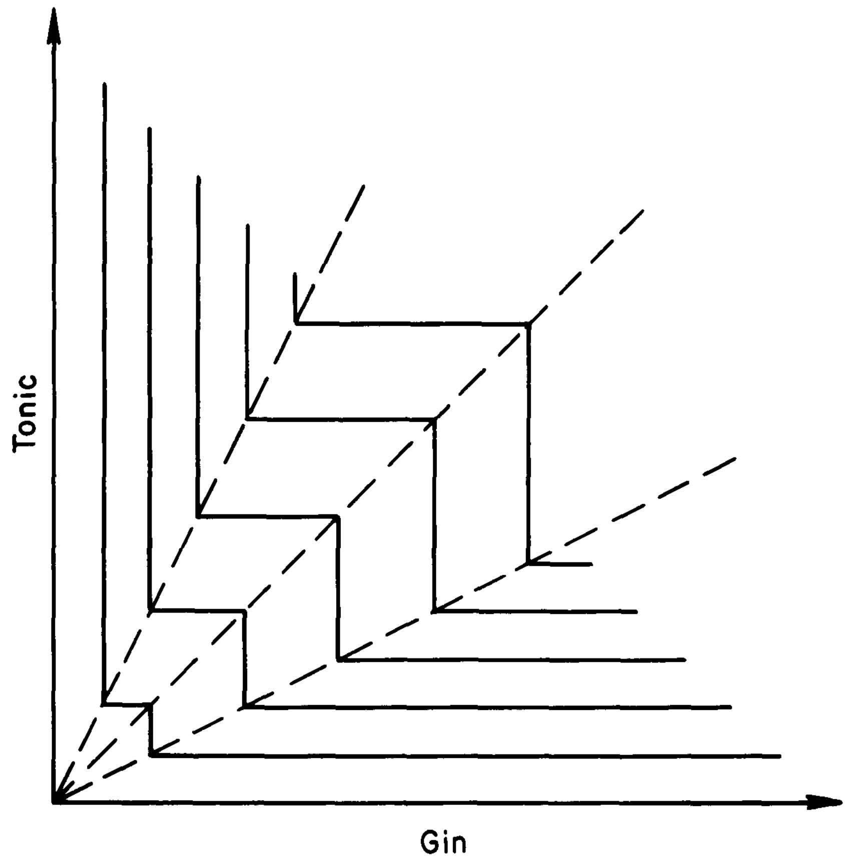

Fig. 1—The commodity space

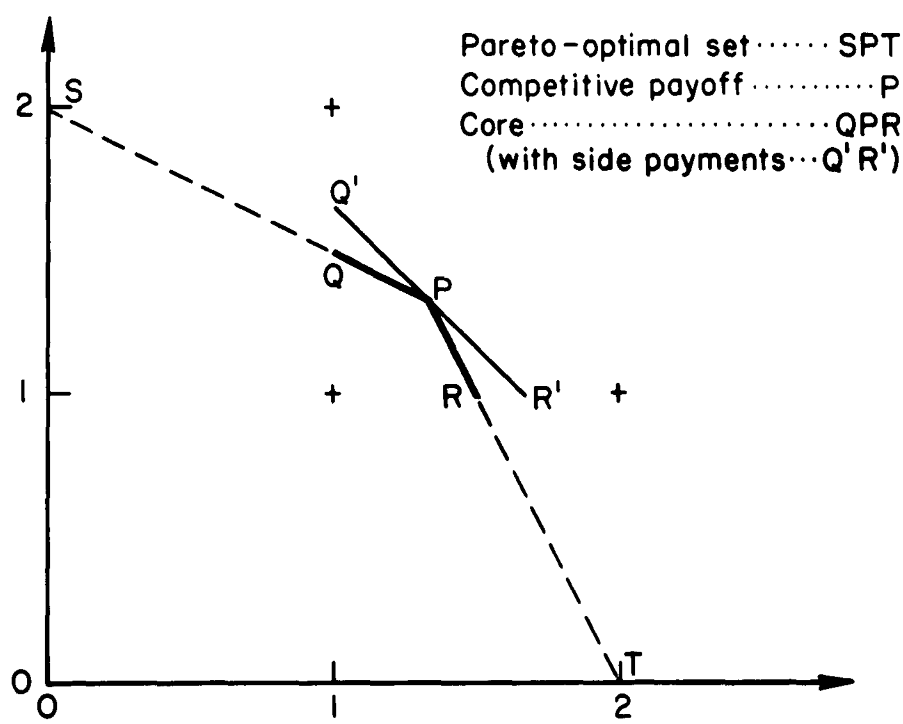

Fig. 2—The utility space

to circumvent the mismatching difficulty previously experienced, and there is no longer any trouble reaching a feasible competitive allocation. (For example, one man can sell out completely, leaving an even number of traders in the market.) The final payoff, as in the even case, will be exactly 4/3 to each trader.

In passing, we may note that if there were diminishing returns to scale in (1), then transferability would no longer "save" the odd case. For example, we might introduce a new utility equal to the square root of the old. Then the equilibrium prices would be (it turns out) precisely $1/\sqrt{12}$ units of utility for each good. What is important is not this number, but the fact that each trader would have just two ways to optimize, neither one involving any net utility payment. In this case, transferability is irrelevant, and there is no competitive allocation.

What of the cores? When $n = 2$, the core (in the utility space) is the bent line QPR, illustrated in Fig. 2.[5] It includes the competitive payoff P, and is included in the set of Pareto-optimal payoffs SPT. If side payments are permitted, the core is instead the straight line Q'R', but it still contains P.

For larger values of n, the recontracting principle takes over with a vengeance: the shrinkage is immediate

[5]In the commodity space, the contract "curve" is rather spectacular. The reader can verify that it consists of four triangular regions, arranged in a ring.

and total. The cores actually contain no points other than the competitive payoffs. That is, when n is odd and side payments are not permitted, there is no core at all; in all other cases the core is a single point. This drastic curtailment of the core is rather atypical, and may be ascribed to the extreme symmetry of the model.[6]

These assertions about the cores are not entirely trivial, and the proofs, which are given in Appendix 1 primarily for the sake of completeness, may also be of interest to the reader unfamiliar with the techniques of core analysis.

4. GAMES AND CORES

The characteristic function of a game ([20], [14]) is designed to express the optimum result or results obtainable by each coalition S of players, regardless of the actions of the players outside S. In the transferable-utility case,[7] it is a real-valued set function $v(S)$, arbitrary except for the condition $v(0) = 0$ and for the property of superadditivity:

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[6]The present model provides no instances of cores without competitive equilibria; such examples, however, are easily constructed.

[7]Without transferable utility, the characteristic function is set-valued, and can be defined in two ways: $v(S)$ represents either (α) the set of payoff vectors that S can surely achieve, or (β) the set that S cannot be preventing from achieving. The two parallel theories that emerge are not generally equivalent because the minimax theorem for coalitions is not valid in the absence of transferable utility. (See Aumann and Peleg [4].)

(2) $\qquad v(S \cup T) \geqq v(S) + v(T)$, if $S \cap T = 0$.

The outcome of the game is conveniently summarized in a payoff vector, α, with components measured in utility units; clearly, we must have

(3) $$\sum_N \alpha_i \leqq v(N),$$

where N denotes the set of all players. A payoff vector is called an imputation it if satisfies two other requirements:

(4) $$\sum_N \alpha_i \geqq v(N)$$

and

(5) $$\alpha_i \geqq v(\{i\}) \text{ for all } i \in N,$$

corresponding respectively to Pareto optimality and individual rationality. These conditions can always be met, because of (2). The more extensive requirement of group rationality:

(6) $$\sum_S \alpha_i \geqq v(S) \quad \text{for all } S \subset N,$$

which includes both (4) and (5), also suggests itself; the set of payoff vectors thereby delimited, if any, is called the core. Since (3) and (6) may well be inconsistent, however, the core does not always exist.

Speaking informally and intuitively, it would appear that a coreless game might be more competitive and harder to stabilize than one with a core. Indeed, in any game, an observed outcome falling outside the core would seem to admit of only two interpretations: (a) it is a transient event in some dynamic process, or (b) it is evidence of a social structure among the players that inhibits certain coalitions from developing their full potential. An observed outcome <u>in</u> the core, on the other hand, tells us nothing at all about the organization of society; the core is sociologically neutral [17].

The mathematical results in this paper depend on the device of enlarging the core by a small amount. Two related concepts will be needed. If ϵ is a small positive number, we define the <u>strong ϵ-core</u> as the set of payoff vectors α satisfying

$$(7) \qquad \sum_S \alpha_i \geqq v(S) - \epsilon, \quad \text{for all } S \subset N,$$

and the <u>weak ϵ-core</u> as the set of payoff vectors satisfying

$$(8) \qquad \sum_S \alpha_i \geqq v(S) - s\epsilon, \quad \text{for all } S \subset N,$$

where s denotes the number of elements of S. It is easy to see that the following relations hold:

$$(9) \qquad \text{weak } \epsilon\text{-core} \supset \text{strong } \epsilon\text{-core} \supset \text{weak } \left(\frac{\epsilon}{n}\right)\text{-core} \supset \text{core},$$

n being the number of players in the game.

These quasi-cores are not just technical devices. For example, they provide a way, if we want one, of accounting for costs of coalition-forming in the solution of the game. Under the weak definition, the costs would be assumed proportional to the size of the coalition, while in the strong case there would be a fixed charge. Alternatively, we may regard the organization costs as included already in the characteristic function, but view the ϵ or $s\epsilon$ as a threshold value, below which the blocking maneuver implicit in (6) —the actual exercise of "group rationality" —is not considered worth the trouble.

5. THE MARKET MODEL

Let there be m different commodities and t_0 different *types* of traders, distinguished by the stocks of goods they hold at the beginning of the trading session. The initial endowment of a player of type "t" will be denoted by a vector

$$a^t = (a_1^t, \ldots, a_m^t).$$

If S is a set of players, and if s_t denotes the number of players in S of type "t", then the aggregate initial endowment of S may be written as follows:

$$a(\mathcal{S}) = \sum_{t=1}^{t_0} s_t a^t,$$

where $\mathscr{s}$ is an abbreviation for the integer vector $(s_1,\ldots,s_{t_0})$ and is called the profile of S. The total supply of goods in the game is then $a(\mathscr{n})$, where $\mathscr{n}$ is the profile of the set N of all players.

At the conclusion of trading, the players hold bundles x^i, which must account for the total quantities initially present in the market. Thus we have

$$(9) \qquad x^i \in E_+^m, \text{ and } \sum_N x^i = a(\mathscr{n}),$$

where E_+^m denotes the closed positive orthant of cartesian m-space. Subject to these constraints, all final allocations are assumed possible. In particular, the outcome need not be symmetric as between players of the same type. In addition, there may be direct transfers of utility among the players. Thus, if U is the common utility measure, the possible final payoffs will take the form

$$\alpha_i = U(x^i) - \pi_i, \quad \text{all } i \in I,$$

subject to (9) and $\Sigma_N \pi_i = 0$.

By symmetry, the characteristic function depends only on the profile of a coalition. Since internal utility transfers will cancel, we have

$$(10) \qquad v_U(\mathscr{s}) = \sup_{y^\nu} \sum_{\nu=1}^{s} U(y^\nu), \text{ subject to } \begin{cases} \text{all } y^\nu \in E_+^m, \\ \Sigma\, y^\nu = a(\mathscr{s}), \end{cases}$$

where s denotes the sum of the s_t. Players outside the coalition do not affect this value, since they can neither force nor be forced into dealings with the coalition members.[8]

Let $\Gamma_U(n)$ designate the game we have just defined. The subscript "U" will serve to distinguish it from an auxilliary game we shall introduce in the next section.

* * * *

An allocation $\{x^{*i}\}$ satisfying (9) is called <u>competitive</u> if there exists a price vector $p = (p_1, \ldots, p_m)$ such that, for each individual $i \in N$, the bundle x^{*i} maximizes the expression:

$$U(x^i) - p \cdot (x^i - a^i). \tag{11}$$

The prices p are called <u>equilibrium prices</u>. (This differs from the ordinary definition, which includes the budget constraints: $p \cdot (x^i - a^i) = 0$. The modification is of course due to the presence of transferable utility.)

Let α_i^* denote the maximum of (11), for a given equilibrium price vector. Then it is easy to show that the vector α_i^*, which we shall refer to as a <u>competitive imputation</u>, is in the core of the game $\Gamma_U(n)$. Indeed, we have

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[8] Hence, questions regarding the proper evaluation of "threats" in the characteristic function do not arise in this game.

$$\sum_N \alpha_i^* = \sum_N U(x^{*i}) \leqq v_U(\mathcal{N}) ,$$

verifying the feasibility requirement (3). To verify the core inequalities (6), we fix $S \subset N$ and $\epsilon > 0$, and, using (10), find bundles $y^i \in E_+^m$ such that

$$\sum_S U(y^i) \geqq v_U(\mathcal{S}) - \epsilon \text{ and } \sum_S y^i = a(\mathcal{S}).$$

Since α^* maximizes, we have

$$\alpha_i^* \geqq U(y^i) - p \cdot (y^i - a^i) \text{ for each } i \in S.$$

Summing, we obtain

$$\sum_S \alpha_i^* \geqq \sum_S U(y^i) \geqq v_U(\mathcal{S}) - \epsilon.$$

Since this is valid for arbitrarily small ϵ, (6) follows, and we have established the following result:

<u>Theorem 1. Every competitive imputation is in the core.</u>

We remark that this result does not depend on our use of identical utility functions for the players, nor have we imposed any regularity requirement on U, apart from the assumption that the game is well defined.[9]

[9] Specifically, U(x) must be locally bounded from above in order for the "sup" in (10) to be finite.

6. "CONCAVIFICATION" OF THE UTILITY FUNCTION

Initially we shall impose almost no conditions on the function $U(x)$. It need not be concave, continuous, or monotonic, and it may either be bounded or unbounded. We do require that its asymptotic growth in E_+^m be no more than linear, and that it be bounded from below on all compact subsets of E_+^m. In other words, we assume the existence of a linear function L_0 and a continuous function K_0 such that the inequalities

$$(12) \qquad K_0(x) \leq U(x) \leq L_0(x)$$

hold for all x in the commodity space E_+^m.

Let us now define a function C on E_+^m as follows:

$$(13) \quad C(x) = \sup \sum_{h=1}^{m+1} \lambda_h U(y^h), \text{ subject to } \begin{cases} \lambda_h \geq 0, \ \Sigma\lambda_h = 1, \\ y^h \in E_+^m, \ \Sigma\lambda_h y^h = x. \end{cases}$$

The finiteness of this "sup" is ensured by (12). The function C is concave, it majorizes U, and it is the least such function.[10] We may remark that C is continuous at every interior point x of E_+^m and possesses a linear support there, i.e., a linear function L such that $L \geq C$ and $L(x) = x$. If the "sup" (13) is actually achieved for all x, we shall say that C is spannable by U (see Sec. 8 below).

[10] The use of m + 1 spanning points is sufficient to "concavify" any linearly bounded function on E^m. Replacing "m + 1" in (13) by a larger number would not affect the value of C(x).

We intend to use C as an artificial utility function in defining a concave majorant game $\Gamma_C(\mathcal{n})$, identical in every other respect to the game $\Gamma_U(\mathcal{n})$ previously defined.[11] The characteristic function v_C has a simple explicit form. Since players have identical, concave utilities, a coalition achieves maximum profit by dividing its total endowment equally among its members, and we have

$$(14) \qquad v_C(\mathcal{s}) = sC(\frac{a(\mathcal{s})}{s}) .$$

Thus, $v_C(\mathcal{s})$, like $a(\mathcal{s})$, is homogeneous of degree one, i.e.,

$$(15) \qquad v_C(k\mathcal{s}) = kv_C(\mathcal{s}), \quad k = 0, 1, 2, \ldots,$$

indicating constant returns to scale in the artificial game to a uniformly expanding coalition. For the true game, on the other hand, we know only that

$$(16) \qquad v_U(k\mathcal{s}) \geq kv_U(\mathcal{s}), \quad k = 0, 1, 2, \ldots$$

—a consequence of superadditivity (2). Our results on the existence of ϵ-cores will hinge on showing that $v_U(\mathcal{s})$ is nevertheless "almost" homogeneous, in a sense to be described.

7. EXISTENCE OF THE WEAK ϵ-CORE

Theorem 2. For every profile $\mathcal{n} = (n_1, \ldots, n_{t_0})$, and for every $\epsilon > 0$, there exists a constant k_0 such that every

[11]Concave utilities, of course, imply convex preference sets.

game $\Gamma_U(k\mathcal{N})$ *with* $k \geqq k_0$ *possesses a weak* ϵ*—core*.

Lemma. (weak ϵ*—homogeneity of* v_U*). For every profile* $\mathcal{S}$ *and for every* $\epsilon > 0$, *there exists a constant* $k_0 = k_0(\mathcal{S},\epsilon)$ *such that*

$$v_C(\mathcal{S}) - \epsilon \leqq \frac{1}{k} v_U(k\mathcal{S}) \leqq v_C(\mathcal{S}) \tag{17}$$

holds for all $k \geqq k_0$.

Proof. Fix $\mathcal{S}$ and ϵ, and let $x^* = a(\mathcal{S})/s$. Then, by (14),

$$v_C(\mathcal{S}) = sC(x^*).$$

Using (13), find a convex representation $x^* = \Sigma\lambda_h y^h$ such that

$$C(x^*) \leqq \sum_{h=1}^{m+1} \lambda_h U(y^h) + \frac{\epsilon}{2s} . \tag{18}$$

Let ℓ_h denote the greatest integer in $ks\lambda_h$, $h = 1,\ldots,m+1$. Then

$$\Sigma\, \ell_h y^h \leqq ks\ \Sigma\lambda_h y^h = ksx^* = a(k\mathcal{S}). \tag{19}$$

Thus, in a coalition with profile $k\mathcal{S}$, it is possible to assign the bundle y^1 to the first ℓ_1 players, y^2 to the next ℓ_2 players, and so on. If "<" holds in (19), there will be goods left over after this allotment, but there will also be at least one player left over, too. Allotting the excess goods equally among the extra players gives

them each an allocation y^* that lies within the convex hull of the $\{y^h\}$. Since there are at most m extra players, we can write down an upper bound for the amount that the allotment of the excess might deduct from the total coalition utility, namely:

$$b = m \cdot |\min_{y^*} K_0(y^*)|, \text{ subject to } y^* \in \text{convex hull of } \{y^h\}.$$

(This is the only use of the function K_0 postulated in (12).) The important fact about this bound is that it is independent of k.

We have thus described a feasible allocation, whose value to the coalition is at least $\Sigma\, \ell_h U(y^h) - b$. Thus

$$v_U(k\mathscr{s}) \geqq \Sigma \ell_h U(y^h) - b.$$

Applying (18), we obtain

$$\begin{aligned} v_U(k\mathscr{s}) &\geqq ks\ \Sigma\lambda_h U(y^h) + \Sigma(\ell_h - ks\lambda_h)U(y^h) - b \\ &\geqq ks\ C(x^*) - \frac{k\epsilon}{2} + \Sigma(\ell_h - ks\lambda_h)U(y^h) - b \\ &\geqq kv_C(\mathscr{s}) - \frac{k\epsilon}{2} - \Sigma|U(y^h)| - b. \\ &= kv_C(\mathscr{s}) - \frac{k\epsilon}{2} - \frac{k_0\epsilon}{2}, \end{aligned}$$

where

$$k_0 = \frac{2}{\epsilon}\,(\Sigma|U(y^h)| + b) \geqq 0.$$

Then $k \geqq k_0$ implies that

$$v_U(k\mathscr{s}) \geqq kv_C(\mathscr{s}) - k\epsilon,$$

giving us one side of (17). The other side is a consequence of (15) and the general inequality $v_U \leq v_C$. This completes the proof of the lemma.

Proof of Theorem 2. Let α be the payoff vector associated with a competitive allocation of the "artificial," concave game $\Gamma_C(\mathcal{n})$. Then α is in the core of that game (Theorem 1). Moreover, for every k, the k-fold replication of α is the payoff vector of a competitive allocation of the game $\Gamma_C(k\mathcal{n})$, and lies in its core. Denote this k-fold replication (a vector with kn components) by α^k. We shall now construct a nearby imputation of the original game $\Gamma_U(k\mathcal{n})$. Denote the difference $v_C(k\mathcal{n}) - v_U(k\mathcal{n})$ by g; clearly $g \geq 0$. Choose an arbitrary n-vector γ whose components sum to g/k and satisfy

$$0 \leqq \gamma_i \leqq \alpha_i - v_U(\dot{\imath}),$$

where $\dot{\imath}$ denotes the profile of the 1-player set $\{i\}$. This is possible because of the two inequalities:

$$g/k \leqq v_C(\mathcal{n}) - v_U(\mathcal{n}) \leqq \Sigma_i(\alpha_i - v_U(\dot{\imath})),$$

and

$$0 \leqq v_C(\dot{\imath}) - v_U(\dot{\imath}) \leqq \alpha_i - v_U(\dot{\imath}).$$

(The first follows from (15), (16), (4) applied to $\Gamma_C(\mathcal{n})$, and the superadditivity of v_U; the second is a consequence

of (5) applied to $\Gamma_C(\mathcal{n})$.) Let γ^k be the k-fold replication of γ, and define

(20) $$\beta^k = \alpha^k - \gamma^k.$$

This is the desired imputation of $\Gamma_U(k\mathcal{n})$. We wish to show that it lies in the weak ϵ-core, in the sense of (8), whenever k is greater than the constant $k_0 = k_0(\mathcal{n}, \epsilon)$ provided by the lemma.

Consider, therefore, an arbitrary subset S of the set of all kn players. Note first that

(21) $$\sum_S \alpha_i^k \geq v_C(\mathscr{s}) \geq v_U(\mathscr{s}),$$

since α^k is in the core of $\Gamma_C(k\mathcal{n})$ and $C \geq U$. Also we have

(22) $$\sum_S \gamma_i^k \leq s\,\frac{g}{k} = sv_C(\mathcal{n}) - \frac{s}{k}\,v_U(k\mathcal{n}) \leq s\epsilon,$$

since each $\gamma_i \leq g/k$ and $k \geq k_0(\mathcal{n}, \epsilon)$. Combining (20), (21), and (22) now gives the desired result:

(23) $$\sum_S \beta_i^k \geq v_U(\mathscr{s}) - s\epsilon.$$

This completes the proof of Theorem 2.

We may observe that $\gamma \to 0$ as $\epsilon \to 0$, so that the weak ϵ-cores can be said in a certain sense to possess a limit point—namely, the imputation α replicated an infinite number of times. Let us state this more precisely. Given any competitive payoff α of the concave game $\Gamma_C(\mathcal{n})$, and

any $\delta > 0$, then for all sufficiently small $\epsilon > 0$ there is an n-vector at a distance less than δ from α whose k-fold replication is in the weak ϵ-core of the game $\Gamma_U(k\mathcal{n})$, for all $k \geq k_0(\mathcal{n}, \epsilon)$. It further appears, in analogy with the results of Debreu and Scarf [7], that the above does not hold for α not a competitive payoff of $\Gamma_C(\mathcal{n})$; we believe, in other words, that the weak ϵ-cores converge (in the sense above) to exactly the set of competitive payoffs of the "concavified" game.

8. FURTHER CONDITIONS ON U; DISCUSSION OF AN EXAMPLE

The extremely weak conditions that we have so far imposed upon the utility function (see Sec. 6) will need reinforcement before a result analogous to Theorem 2 can be obtained for strong ϵ-cores; we shall require the "spannability" of C (see Sec. 6) and a certain amount of differentiability for U. Our primary purpose, of course, is the study of the effects of nonconcavity in U, not the wholesale abandonment of regularity assumptions of all kinds. Our policy of keeping the hypotheses as general as possible serves a secondary function, by making clear precisely what our results do and do not depend upon.

We shall postulate, then, that U is radially differentiable, and has a spannable concave majorant. The first assumption means that U is differentiable along all rays in E^m_+ emanating from the origin.[12] This is

[12] Actually, we shall use the radial derivatives only at points where U and C are equal.

considerably weaker (if $m > 1$) than assuming the existence of first partial derivatives $\partial U/\partial x_i$. We do not demand that the radial derivatives be continuous, or, indeed, that U itself be continuous.

The second assumption means that there exists a concave function $C \geqq U$ such that for each $x \in E^m_+$, there are m+1 (or fewer) points $y^h \in E^m_+$ and weights $\lambda_h \geq 0$ such that

$$\Sigma \lambda_h = 1, \quad \Sigma \lambda_h y^h = x, \text{ and } \Sigma \lambda_h U(y^h) = C(x).$$

This clearly implies that U is bounded above by a linear function, as previously assumed (right side of (12)). The matching assumption that U is bounded below by a continuous function (left side of (12)) is not implied, and is not needed.

Since the notion of spannability, which may be an unfamiliar one to many readers, is evidently quite fundamental to any investigation of the relaxation of convexity conditions, a short digression is now in order, to link this notion to other analytic conditions and to explicate its rôle in our present work. Accordingly, in Theorem 3 and the remarks following, we shall indicate some sufficient conditions for spannability, and then work through a simple group of examples.

Let U be called <u>sublinear</u> if for every linear function L with positive coefficients, the difference U—L has an

upper bound. For example, logarithmic (Bernoullian) utility is sublinear; also, any bounded utility function is sublinear.

Theorem 3. If U is continuous, sublinear, and strictly increasing, then U has a spannable concave majorant C.

From the proof (given in Appendix 2) it will be seen that "strictly increasing" is hardly necessary here; a certain very weak insatiability condition would suffice instead, namely: for each $x \in E_+^m$ and each $j = 1,2,\ldots,m$, there is a $y \in E_+^m$ differing from x only in the j–th component such that $y_j > x_j$ and $U(y) > U(x)$. In addition we may remark that if U has a spannable concave majorant, then so does $U + L$, for any linear function L, despite the fact that the addition of L may destroy both sublinearity and monotonicity (or insatiability).

Let us now consider a very simple example, having just one commodity and one type of player, in order to show how spannability and differentiability are crucial to the existence of strong ϵ–cores in the limit. Let $U(x) = [x/2]$, i.e., the greatest integer less than or equal to $x/2$, and let all the initial endowments be 1 unit. Here $C(x) \equiv x/2$, and is spannable, but U is not differentiable. (See Fig. 3a.) For a coalition with s members we have

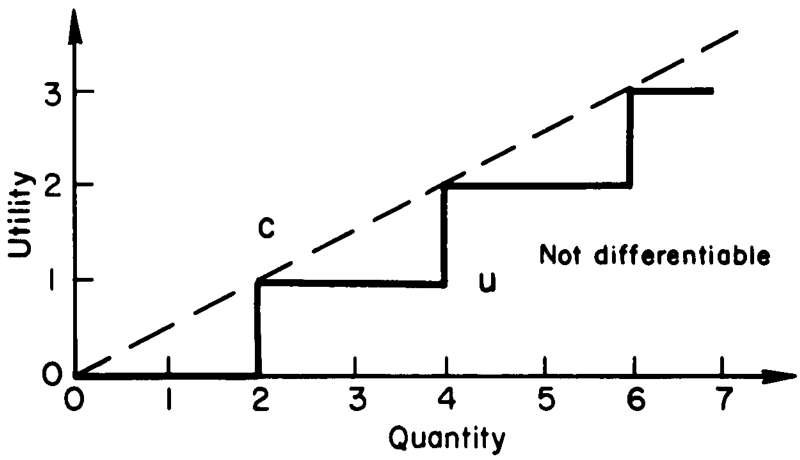

Fig. 3a

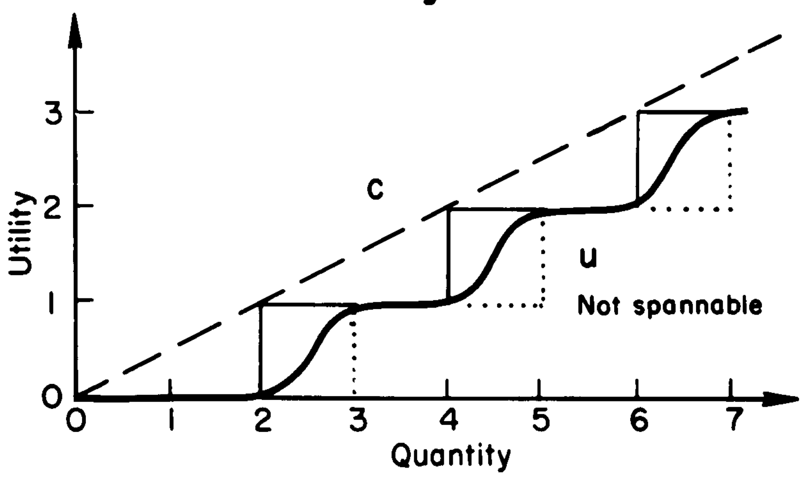

Fig. 3b

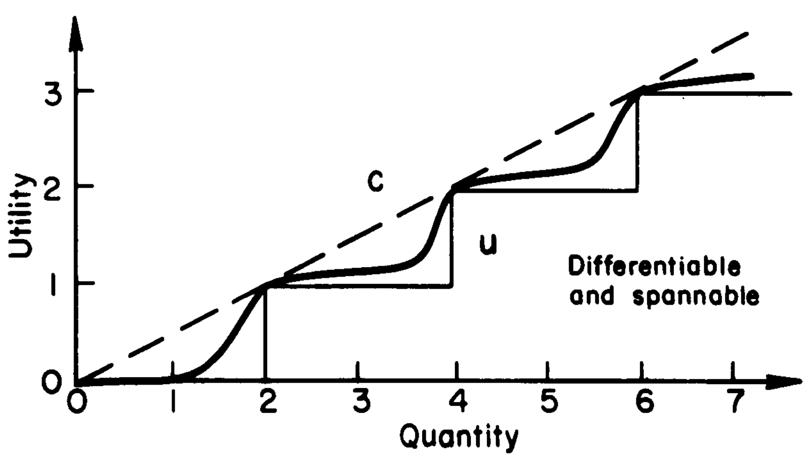

Fig. 3c

$$v_U(s) = [s/2] \text{ and } v_C(s) = s/2.$$

If the game happens to have an odd number of players, say $n = 2r + 1$, then in any Pareto-optimal payoff vector, the most-favored player receives at least r/n units. The $2r$ least-favored players therefore receive at most $r - r/n$. This must be compared with the amount r that they can obtain in coalition. The difference, r/n, converges not to 0, but to 1/2 as $r \to \infty$. Hence strong ϵ-cores do not exist for large, odd n and small ϵ.

Now let us change U to destroy the spannability of C, at the same time making U differentiable (Fig. 3b).[13] One can then verify that

$$(24) \qquad v_U(s) = U(s) = [\tfrac{s-1}{2}], \; s = 1, 2, \ldots .$$

In other words, a coalition can do no better than allot all its goods to one player.[14] If there happens to be an even number of players in the game, say $n = 2r$, then the least-favored set of $n-1$ players will always get $(r-1) - (r-1)/n$ or less, compared with the amount $r-1$ they can obtain in coalition. Again, the difference goes to 1/2 as $r \to \infty$,

[13] The precise form of the curved parts of U within the small squares is immaterial.

[14] If the coalition allots in integer units, U might as well be the lower step-function in the figure (dotted lines). If fractional shares are used, the result can be no better than discarding one unit and then using the upper step-function. In either case the value (24) is best possible.

and strong ϵ—cores do not exist for large r and small ϵ.

Finally, let us restore spannability, as in Fig. 3c, taking care to make U differentiable at the points of contact with C. A coalition with $n = 2r + 1$ members will now be able to allot $2 + 1/r$ units to r of its members and nothing to the other $r+1$ members, receiving a total utility of $r(1 + 1/2r - 0(1/r^2))$. Thus we have

$$v_U(n) = n/2 - 0(1/n)$$

—valid for even as well as odd n. In an n—person game, then, the imputation that gives an equal amount to each player assigns to every s—player set an amount

$$(s/n)v_U(n) = s/2 - 0(1/n) ,$$

compared with at most $s/2$ that they can obtain in coalition. In this case the difference does go to zero, and we have strong ϵ—cores in the limit as $n \to \infty$, for arbitrarily small ϵ.

9. EXISTENCE OF THE STRONG ϵ—CORE

Theorem 4. If U is radially differentiable and has a spannable concave majorant C, then for every profile $\mathcal{n}$ and for every $\epsilon > 0$, there is a constant k_0 such that every game $\Gamma_U(k\mathcal{n})$ with $k \geq k_0$ has a strong ϵ—core.

Lemma. For every profile $\mathcal{s}$ and for every $\epsilon > 0$ there exists a constant k_0 such that

(25) $$v_C(\mathscr{s}) - \frac{\epsilon}{k} \leqq \frac{1}{k}\, v_U(k\mathscr{s}) \leqq v_C(\mathscr{s})$$

<u>holds for all</u> $k \geqq k_0$.

<u>Proof</u>. Fix $\mathscr{s}$ and ϵ, and let $x^* = a(\mathscr{s})/s$. Then

(26) $$v_C(\mathscr{s}) = sC(x^*).$$

Using the spannability of C, find a convex representation $x^* = \Sigma\, \lambda_h y^h$ such that

$$C(x^*) = \Sigma \lambda_h U(y^h).$$

Given k, we wish to "move" the points y^h slightly, to make the coefficients λ_h integer multiples of 1/ks. The technical details of this maneuver have been relegated to Appendix 3; the result is a new convex representation $x^* = \Sigma\, \mu_h z^h$ for each k, with the property that for each h, $ks\mu_h$ is an integer, and

(27) $$ks|U(z^h) - L^*(z^h)| \to 0 \text{ as } k \to \infty,$$

where L^* is the linear function tangent to C at x^*. We therefore have

$$v_U(k\mathscr{s}) \geqq \Sigma\, ks\mu_h U(z^h),$$

since the $ks\mu_h$ are integers with sum ks. Using (27), we have for sufficiently large k

$$v_U(k\mathscr{A}) \geqq ks\Sigma\mu_h L^*(z^h) - \epsilon.$$

The right side is equal to $ksL^*(x^*) - \epsilon$, by linearity. This in turn is equal to $ksC(x^*) - \epsilon$. Applying (26) gives the desired result:

$$v_U(k\mathscr{A}) \geqq kv_C(\mathscr{A}) - \epsilon.$$

The other inequality in (25) is immediate, by (15). This completes the proof of the lemma.

The proof of Theorem 4 proceeds exactly like that of Theorem 2 in Sec. 7, but uses the more powerful lemma that we have established. Note that in order to ensure that the constructed imputation β^k is in the strong ϵ-core, we must use "ϵ/n" for "ϵ" in applying the lemma. The last line of the proof (compare (23) in Sec. 7) then reads as follows:

$$\sum_S \beta_i^k \geqq v_U(\mathscr{A}) - \frac{s\epsilon}{kn} \geqq v_U(\mathscr{A}) - \epsilon.$$

The remarks at the end of Sec. 7 concerning the limiting behavior of weak ϵ-cores apply equally to the present case.

10. CONCLUSION

The core is a more general economic concept than the competitive allocation, in that it is possible to have markets with a core but no competitive equilibrium, but

not vice versa. This can happen whether the absence of a competitive equilibrium is due to nonconvex preferences or to other factors not considered in this paper, such as nonconvex production sets or external production economies. Although no pure price mechanism will clear the market, there may nevertheless exist an imputation of wealth—a core payoff—that is stable against both individual and joint action by the participants. Achieving this imputation may involve taxes, transfer payments, or multiple prices.

There are also cases of interest where the core itself may be vacuous. This means, in terms of Edgeworth's mechanism, that there will always be a group wishing to recontract. To resolve the inherent instability of this situation, we must resort to introducing social, cultural, or institutional restraints. The theory of games offers several solution concepts that provide frameworks for the systematic introduction of these factors; among the more useful are the von Neumann-Morgenstern "solutions" [20], the Luce "ψ-stable pairs" [14], and the Aumann-Maschler "bargaining sets" [3].

Another tool that seems promising in this connection is the quasi-core concept employed in the present paper. It can easily be shown that ϵ-cores (both weak and strong) always exist if ϵ is large enough. The sociological factor involved here can be interpreted as an organizational

cost prerequisite to cooperative action, proportional to the parameter ϵ. Theorems 2 and 4 of this paper indicate that even if ϵ is small, the quasi-cores will exist when the market is large enough. Of course, when ϵ-cores exist for small values of ϵ, it is not unlikely that the core itself exists as well, making the market fully stable against recontracting. But even without a true core, the profit to be gained from recontracting out of an ϵ-core would be small, and a near-stability can be achieved.

By passing to the limit with these quasi-cores, or by direct concavification of the finite model, one can define a price structure that takes the place of the missing competitive equilibrium. These pseudo-equilibria might repay further study. Perhaps (for example) one could find cases where they serve as the limit points of tâtonnement processes ([19]), in the absence of true equilibria.

Appendix 1

DETERMINATION OF CORES IN THE EXAMPLE OF SECTION 3

Generalizing (1), let the utility function be given by

$$(28) \qquad U(x) = \max[\min(ax_1, x_2), \min(x_1, ax_2)],$$

where $a > 1$. Let $A = 2a/(1 + a)$, and note that $a > A > 1$. Suppose first that side payments are permitted. Then it is easily verified that any set of s players, $2 \leq s \leq n$, can achieve a combined payoff of sA, and no more. Thus, the payoff vector P that assigns A to each player is undominated, and lies in the core. Any other payoff vector will assign <u>less</u> than $(n-1)A$ to the $n-1$ least-favored players; hence, if $n \geq 3$, that set of players can block it. Thus the core consists of the single point P. When $n = 2$, however, blocking only occurs if one player is assigned less than 1, or if the two together get less than $2A$. Thus, the core is the line segment joining the points $(1, 2A-1)$ and $(2A-1, 1)$ (Q' and R' in Fig. 2).

Next, let side payments be prohibited, and let $\alpha = (\alpha_1,\ldots,\alpha_n)$ be an undominated payoff vector. We may assume that

$$(29) \qquad \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n .$$

Now, the coalition $\{1,2\}$ can divide its assets as follows:

$$(t,\ t/a) \text{ to } 1,\ (2-t,\ 2-t/a) \text{ to } 2,$$

and thereby achieve the payoffs $\beta_1 = t$, $\beta_2 = 2 - t/a$ for any t between 0 and A (i.e., any point on the line from S to P in Fig. 2). To avoid domination of α by such β, we must have

$$(30) \qquad \alpha_1 \geqq 2a - a\alpha_2 .$$

Combined with (29), this entails

$$(31) \qquad \alpha_2 \geqq \frac{2a}{a+1} = A.$$

We also have, of course,

$$(32) \qquad \alpha_1 \geqq 1.$$

Let $\{(x_1^i,\ x_2^i)\}$ be an allocation that yields α. Let p be the number of indices $i > 1$ such that $x_1^i \leqq x_2^i$, and let $q = n - 1 - p$. We note from (28) that $x_1^i \leqq x_2^i$ implies that $\alpha_i = \min(ax_1^i,\ x_2^i)$, and that $x_1^i > x_2^i$ implies that $\alpha_i = \min(x_1^i,\ ax_2^i)$. Hence

$$p\alpha_2/a + q\alpha_2 \leqq \sum_2^n x_1^i = n - x_1^1 ,$$

by (29). Hence

$$x_1^1 \leqq n - (p/a + q)\ \alpha_2 = B_1 .$$

Similarly

$$x_2^1 \leqq n - (p + q/a)\,\alpha_2 = B_2.$$

Without loss of generality, we can assume that $p \geqq q$. Then $B_1 \geqq B_2$, and we have

(33) $$\alpha_1 \leqq \min(B_1, aB_2).$$

<u>Case i</u>. Suppose $p = q$. Then $n = 2p + 1$, and

(34)
$$\begin{aligned} 1 \leqq \alpha_1 \leqq B_1 &= (2p + 1) - p(\tfrac{1}{a} + 1)\alpha_2 && \text{(by (32) and (33))} \\ &\leqq 2_p + 1 - p\,\frac{a + 1}{a}\,A && \text{(by (31))} \\ &= 1. \end{aligned}$$

Hence there is equality throughout (34), and $\alpha_1 = 1$ and $\alpha_2 = A$. But this violates (30). Hence we cannot have $p = q$.

<u>Case ii.</u> Suppose $p \geqq q + 2$. Then

$$\begin{aligned} 1 \leqq \alpha_1 \leqq aB_2 &= (p + q + 1)a - (pa + q)\alpha_2 && \text{(by (32) and (33))} \\ &\leqq a + p(a - aA) + q(a - A) && \text{(by (31))} \\ &\leqq a + p(a - aA) + (p - 2)(a - A) = 2A - a \\ &= 1 - \frac{(a - 1)^2}{a + 1} < 1. \end{aligned}$$

Hence we cannot have $p \geq q + 2$. There remains...

Case iii. Suppose $p = q + 1$. Then $n = 2p$, and we have

$$2a - a\alpha_2 \leq \alpha_1 \leq aB_2 = 2pa - (pa + p - 1)\alpha_2,$$

from (30) and (33). Hence

$$(35) \qquad (p - 1)(a + 1)\alpha_2 \leq 2(p-1)a.$$

Now if $p > 1$, this gives $\alpha_2 \leq A$; hence, by (31), $\alpha_2 = A$, and we have equality in (35). But in deriving (35) we made essential use of (30) and most of (29); therefore, equality must prevail in these places as well, and we find that all the α_i are equal to A. In other words, the only possible candidate for the core is the vector $\alpha = (A,A,\dots,A)$. It is easily verified that this vector is in fact a feasible payoff, and is undominated; hence we have a one-point core as claimed. Note that n in this case is even.

If $p = 1$(i.e., $n = 2$), then (35) is no restriction, and it is easy to show that the core consists of the broken line from $(1, 2 - \frac{1}{a})$ to (A, A) to $(2 - \frac{1}{a}, 1)$ (QPR in Fig. 2).

Appendix 2

PROOF OF THEOREM 3

Define C as in (13), take x^* interior to E_+^m, and let L be a linear support to C at x^*. Then L is strictly increasing in each x_j, and so is $L/2$. By sublinearity, we can find a function L' parallel to $L/2$ such that $L' \geq U + c$, where c is a preassigned positive constant. Let R denote the region of E_+^m in which $L \leq L'$; clearly R is compact and contains x^*. We now wish to consider convex representations of x^* that "almost" achieve the value $C(x^*)$, in the sense of the "sup" in (13). In order to distinguish between vertices lying within R and those outside, the representations will be written in the following way:

$$x^* = \alpha y + \bar{\alpha} z = \alpha \Sigma \lambda_h y^h + \bar{\alpha} \Sigma \mu_k z^k, \; y^h \in R, \; z^k \notin R. \tag{36}$$

Here $\bar{\alpha}$ denotes $1 - \alpha$, and is understood to be 0 if there are not points of the second type in the representation, i.e., if z is not well-defined. (Note that y is always well-defined; this follows from the fact that x is outside the convex set $E_+^m - R$.) Given $\epsilon > 0$, by (13) we can find a representation satisfying

$$\alpha \Sigma \lambda_h U(y^h) + \bar{\alpha} \, \Sigma \mu_k U(z^k) \geq C(x^*) - \epsilon.$$

Hence we have

$$\alpha L(y) + \bar{\alpha}L'(z) - \bar{\alpha}c \geq L(x^*) - \epsilon,$$

or, from (36) and the linearity of L,

$$\epsilon \geq \bar{\alpha}\ [L(z) - L'(z) + c].$$

Since $[L(z) - L'(z) + c] \geq c > 0$ for $z \in E^m_+ - R$, we see that

$$\bar{\alpha} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

But since L' and $L/2$ are parallel, the expression in brackets is of the form $L'(z) + c'$, with a new constant c'. Thus, even though $\|z\|$ may be unbounded, we nevertheless have

$$\bar{\alpha}\ \|z\| \to 0 \quad \text{and} \quad \bar{\alpha}\ L'(z) \to 0, \text{ as } \epsilon \to 0.$$

Hence,

$$y = \frac{1}{\alpha}\ (z^* - \bar{\alpha}z) \to x^*, \text{ as } \epsilon \to 0,$$

and

$$\begin{aligned} \Sigma\lambda_h U(y^h) &\geq \frac{1}{\alpha}\ [C(x^*) - \epsilon - \bar{\alpha}\ \Sigma\mu_k U(z^k)] \\ &\geq \frac{1}{\alpha}\ [C(x^*) - \epsilon - \bar{\alpha}\ L'(z)] \\ &\to C(x^*). \end{aligned}$$

Since the y^h are restricted to the compact region R and since U is assumed to be continuous, there exists a limiting representation $x^* = \Sigma\lambda_h y^h$ with $\Sigma\lambda_h U(y^h) = C(x^*)$.

Hence C is spannable by U at the arbitrary interior point x^*.

If x^* is not interior to E_+^m, this argument is not directly valid, since no linear support L need exist. But we can then reduce the dimension of the problem, without affecting either the definition of $C(x^*)$ or the hypotheses of continuity, sublinearity, and strict monotonicity. In the reduced problem, x^* will be interior.

Appendix 3

PROOF OF (27) IN SECTION 9

Lemma. Let L^* be a support to C at x^*. Let there be a convex representation $x^* = \Sigma\lambda_h y^h$ such that $\Sigma\lambda_h U(y^h) = C(x^*)$. Assume that U possesses a radial derivative at each y^h. Let s be a fixed integer and k a variable integer. Then there exist convex representations $x^* = \Sigma\mu_h z^h$, depending on k, such that for each h, $ks\mu_h$ is an integer and

$$(37) \qquad ks|U(z^h) - L^*(z^h)| \to 0, \text{as } k \to \infty.$$

Proof. Assume first that all λ_h are positive. For any k, we can find nonnegative integers ℓ_h with sum ks and such that $|ks\lambda_h - \ell_h| \leqq 1$. Now define:

$$\mu_h = \ell_h/ks \quad \text{and} \quad z^h = \frac{\lambda_h}{\mu_h} y^h.$$

For k sufficiently large, the μ_h will be positive, and all the statements in the lemma are obviously satisfied, except for (37). To verify the latter, note that L^* is tangent to U at each y^h, and that z^h approaches y^h along the ray $0y^h$. (If $y^h = 0$ and there is no ray then (37) is trivial.) Since the derivative of U at y^h exists along that ray, we have

$$\frac{|L^*(z^h) - U(z^h)|}{\|z^h - y^h\|} \to 0, \text{as } k \to \infty .$$

(We are concerned, of course, only with values of k for which $z^h \neq y^h$.) But, for large k,

$$\|z^h - y^h\| = \left|\frac{\lambda_h - \mu_h}{\mu_h}\right| \cdot \|y^h\|$$

$$\leqq \frac{1/ks}{\lambda_h - 1/ks} \quad \|y^h\| .$$

Hence $ks\,\|z^h - y^h\|$ goes to zero, and (37) (which is the same as (27)) follows.

If some of the λ_h are zero, we can set the corresponding $\mu_h = 0$ and $z^h = y^h$, and proceed as above, with h restricted to the indices for which λ_h is positive.

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